

Descartes' Rule of Signs

Establishes Descartes' rule of signs

Statement: If $f(x) = 0$, then

- (i) The number of positive roots cannot exceed the number of changes of sign in $f(x)$
- (ii) The number of negative roots cannot exceed the number of changes of sign in $f(-x)$.

Proof - (i) For positive roots:

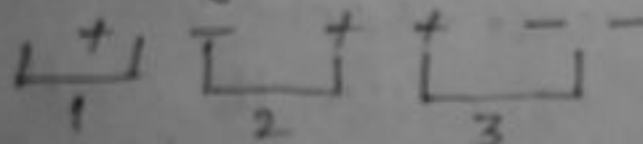
Case I - When polynomial equation cannot have more positive roots than there are changes of signs from + to - and from - to +, in its terms of the corresponding expression

This can be established by the following.

example

$$3x^5 - 7x^4 + 3x^3 + 2x^2 - 8x - 1 = 0$$

The signs of the polynomial are



Here three changes of sign be seen without any loss of generality, let us suppose that the sign of the polynomial expression are

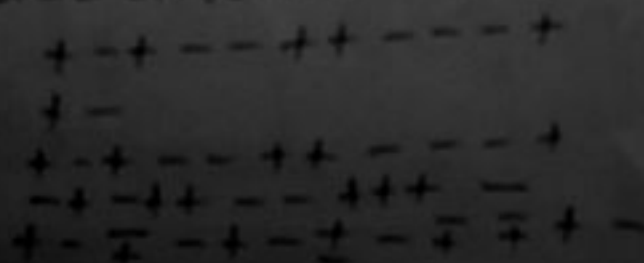
+ - + - - + + - - - +

Clearly the given polynomial has six changes of signs.

Now, we multiply the polynomial by a binomial $x-a$ corresponding to a positive root a .

The signs of the binomial $x-a$ are

Here multiplication can be exhibited as follows:



Now, we are going to show that the resulting polynomial has at least one more change of sign than the original number of changes of signs.

Here we observe 4 ambiguous signs (roots) in the resulting polynomial and so it will be $2^4 = 16$ combinations of these ambiguous signs.

So in this case, the signs of the resulting polynomial are

+ - + - - + + - - - + -

This gives seven changes of signs - one more than the number of changes of signs in the original.

Case II: In this case the number of changes of signs may be even greater than seven. For this take the first ambiguous sign with a + sign second with a - sign, third with a + sign and fourth sign with a - sign. We shall, then get

+ - + - + + - - + - + -

This gives nine changes of signs. So in every case there is at least one variation of sign added.

Thus when the complete product is formed by successively multiplying the polynomial by $x - \alpha$, $x - \beta$, $x - \gamma$ etc, the resulting polynomial will have at least as many changes of signs as it has +ve roots.

(ii) Negative roots: - We know that negative roots of $f(x) = 0$ are the positive roots of $f(-x) = 0$. So from Descartes's rule of signs for positive roots that the number of positive roots of $f(-x) = 0$, cannot exceed the number of changes of signs in the terms of the polynomial $f(-x)$.

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